Portfolio Optimization using Quantum Annealing

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Introduction and Motivation

Markowitz Portfolio Optimization

- **Objective:** Select a portfolio that strikes an optimal balance of risk and return with regard to the investor's risk aversion [1].
- **Portfolio:** The portfolio composition (p) is defined by a weight vector that assigns a weight to each asset: $p = [0.1, 0.2, 0.05, \dots, 0]$, $p_i \in [0,1]$
- **Return:** The portfolio return is defined by the relative change in value over time, where r_i are the returns of the assets to select from:

$$r(p) = \sum_i p_i\, r_i$$

- Risk: The portfolio risk is defined by the variance of the portfolio return:

$$\sigma^2(r|p) = \sum_i \sum_j p_i\, p_j (r_i - \bar{r}_i)(r_j - \bar{r}_j)$$

- Optimization:

$$\min_p (\sigma^2(r|p) - r(p))$$

Quantum Annealing (QA)

- Quantum annealing can solve a binary quadratic optimization problem approximately [2].
- This is done by measuring the lowest energy state with respect to the following Hamiltonian [3]:

$$\min \sum_{i>j} J_{i,j} \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)} + \sum_i h_i \hat{\sigma}_z^{(i)} \qquad \begin{array}{l} J_{i,j} \coloneqq \text{Coupling strength between qubits } i \text{ and } j \\ h_i \coloneqq \text{Bias term of qubit } i \end{array}$$

- QUBO formulation:

$$\min p \begin{pmatrix} h_i & \cdots & J_{1,n} \\ \vdots & \ddots & \vdots \\ J_{n,1} & \cdots & h_n \end{pmatrix} p^T$$

Motivation

Absence of analytical solutions:

- The general case of the Markowitz portfolio optimization shown above has an analytical solution. The drawback of this solution is that it often includes (large) negative values. Negative portfolio positions are restricted by law in some cases, and they carry more risk.
- It is a common constraint that a portfolio should not include negative positions $(p < 0)$.
- Adding this constraint makes the problem analytically insolvable and therefore a good use case for QA.

Real-world applicability:

The focus of this work is to assess the real-world applicability to answer:

- How large can portfolios problems be so that they are still implementable on the annealer?
- Is the solution quality comparable to classical methods and under which conditions?
- Is it economically beneficial to use QA in a portfolio management strategy?

Portfolio Optimization with QA

Portfolio Problem Formulation

- **Challenges:** Mapping the continuous problem to a binary optimization problem causes that the solution vector obtained from the QA only decides over buying or not buying an asset.
- The QUBO of the binary optimization performed by the annealer is given by:

$$\min_p p \begin{pmatrix} \sigma_1^2 - r_1 & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_n^2 - r_n \end{pmatrix} p^T$$

- The diagonal entries consist of the variance of an asset minus its return and the off-diagonal entries representing the covariances between those assets. For simplicity, the risk aversion is not considered.
- The resulting QUBO that is implemented on the annealer represents in nearly every case a fully connected graph. This limits the implementable problem size and increases the chain length of logical qubits [4].

Solving Procedure

After a defined number of annealing procedures, the annealer SDK returns a summary of the results, shown in Table 1.

The complete set of results are then used to determine a relative weight vector in two steps:

1. Every qubit corresponds to one asset. The assets weight p_i is given by the weighted sum of all sample values for this qubit.
2. The resulting vector p is normalized to 1.

The two post-processing steps leads to an improvement in the solution quality by 16%.

	Qubit-Index																			E	o
	0	1	2	3	4	5	6	7	8	9	...	20	21	22	23	24	25	26	27	Energy	Count
0	1	1	1	1	0	0	0	0	0	0	...	0	0	0	0	1	1	1	1	-15.886410	14
1	1	1	1	1	1	0	0	0	0	0	...	0	0	0	0	1	1	1	1	-15.798869	75
2	1	1	1	1	1	0	0	1	0	0	...	0	0	0	0	1	1	1	1	-15.679749	49

Table 1: Result summary of the annealing procedures

$$p_i = \sum_j q_i \left(\epsilon_j\, o_j\right)^\beta \qquad \epsilon_j = \frac{e^{-E_j/kT}}{\sum_l e^{-E_l/kT}} \qquad o_j = \frac{o_j}{\sum_l o_l}$$

$o_j \coloneqq$ number of occurrences
$E_j \coloneqq$ energy of state j
$\beta \coloneqq$ tunable hyperparameter

Trading Simulation

To gain insights on the applicability of QA for portfolio optimization in a real-world setting, it will be applied in a continuous portfolio selection process.

The Simulation visualized in Fig. 1 consists of two entities interacting with each other:

- **The Agent** estimates future returns and risks and optimizes the corresponding portfolio problem. The difference of the solution and the current portfolio is compensated by orders to the market [5].
- **The Market** executes orders using the end-of-day price and charges transaction costs [6].

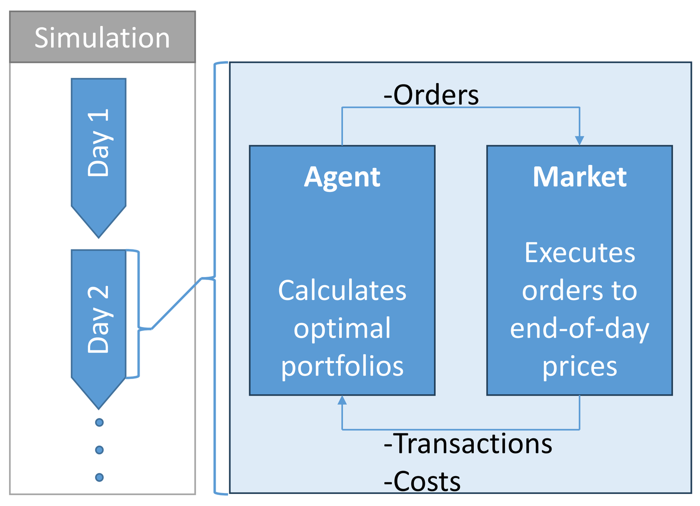

Figure 1: Concept of the trading simulation, which is executed day-by-day with constant rebalancing of the portfolio positions.

Experimental Results

Simulation Results

- Figure 2 shows the wealth accumulation over time for one strategy based on a return estimation model with different risk aversions and different solving methods.
- In the cases QA was used to optimize the portfolio a significantly higher return is visible
- This observation only holds true if the underlying strategy is able to estimate future returns, it completely vanishes in the case for unpredictable returns (for example a pandemic).
- By testing several estimation models with differences in complexity and therefore in predictive power it can be shown that the QA solving must be paired with a strong model to achieve an economic advantage.
- Lower risk aversions lead to better returns only if the conditions from the previous point are met.

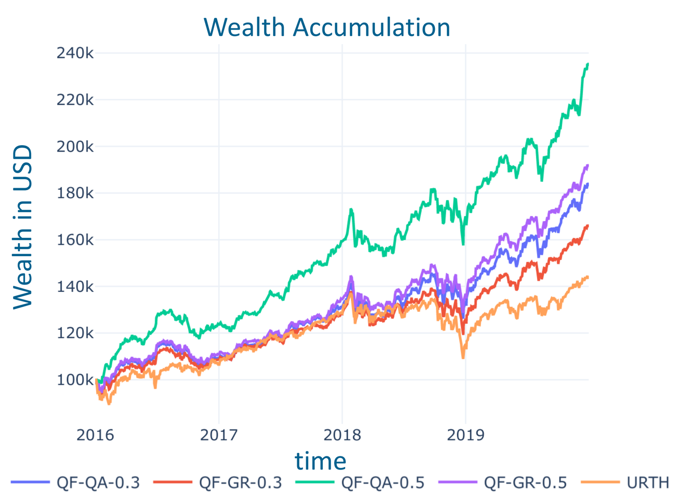

Figure 2: Wealth accumulation using a problem size of 55, different risk aversions and solving methods. QF: quality factor prediction model, QA: quantum annealing solver, GR: gradient solver, URTH: baseline.

Solution Quality

- Figure 3 shows the relative and absolute solution quality of the simulation in figure 2. The change of wealth for the upcoming 180 days is shown as a reference.
- The solution quality of the QA-based method is slightly better than the gradient based optimization. The difference ranges between 5% and 15%.
- The points after 2018, where the gap strongly increases, show a gap in future returns as well.
- For smaller portfolio sizes of 40 assets and for the largest possible number of 70, the advantage was smaller compared to a medium sized portfolio of 55 assets.
- Problems that are dominated by the interaction terms $(J_{i,j})$ show the smallest advantages.

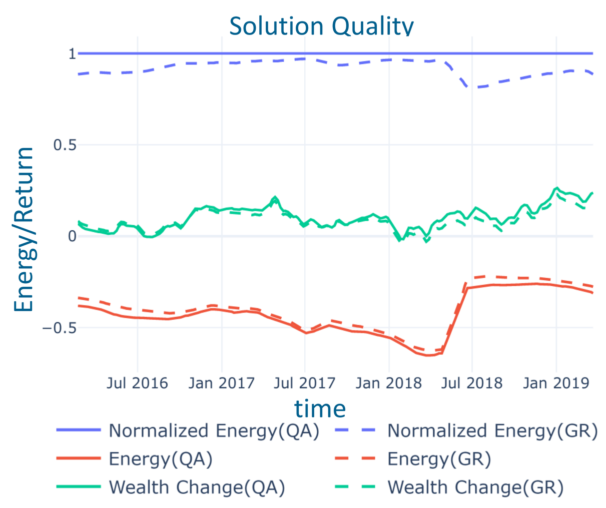

Figure 3: Normalized and absolute energy for different solvers. Future return as comparison.

Solution Characteristics

- Figure 4 shows the entropy of the solutions found by different optimizers paired with different risk aversions, where entropy is defined as: $\text{entropy}(p) = -\sum p_i \log_2(p_i)$
- Lower entropies correspond to more concentrated portfolios that invest in fewer stocks, whereas higher entropies translate to higher diversification.
- It becomes visible that the solutions from the annealer behave different, having lower entropies previous to the corona pandemic (2020 and earlier) and higher after the stock market shock induced by the pandemic (2020, April).
- This might be due to a strong increase in variances and covariances, elevating the energy landscape of the optimization problem.

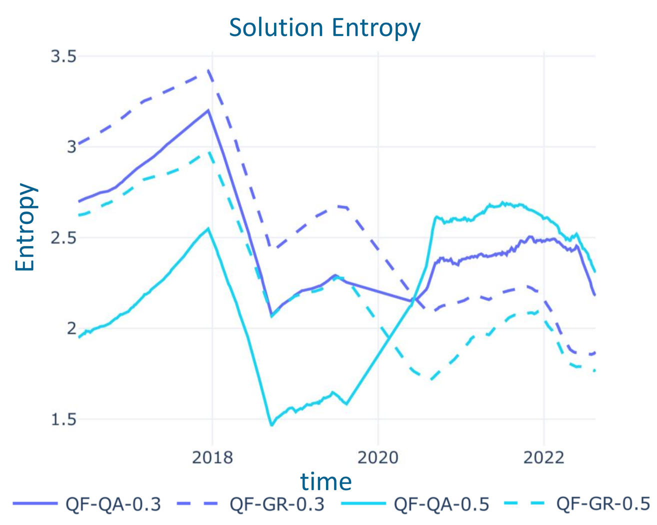

Figure 4: Entropy of solution vectors for the different solving methods.

Conclusion

- QA devices of the current generation are suitable for application in real-world portfolio management.
- The simulations show higher returns during periods where the solution quality of QA is better compared to the classical method.
- The post-processing method is a heuristic with no known theoretical guarantees. However, it worked well for the cases we tested.
- Outlook: Besides the presented method of post-processing that generates a relative weight vector, one can encode a similar behavior into the problem by using multiple qubits per asset.

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The authors gratefully acknowledge the Jülich Supercomputing Centre (https://www.fz-juelich.de/ias/jsc) for funding this project by providing computing time on the D-Wave Advantage™ System JUPSI through the Jülich UNified Infrastructure for Quantum computing (JUNIQ).

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